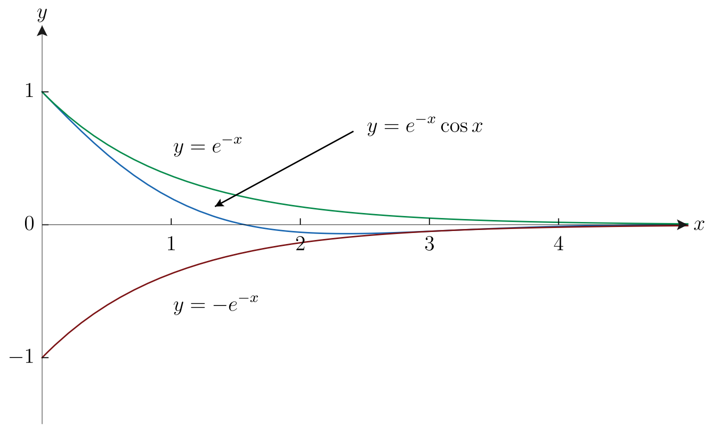
Monday Night Calculus, September 13, 2021

1. $\lim\limits_{x\to\infty} e^{-x}\cos x$

$$\lim_{x\to\infty} \frac{\cos x}{e^x} =$$

$$-\frac{1}{e^x} \le \frac{\cos x}{e^x} \le \frac{1}{e^x}$$

$$\lim_{x\to\infty} -\frac{1}{e^x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{e^x} = 0 \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\cos x}{e^x} = 0$$



$$\lim_{\theta\to 3} \sqrt{\theta - 3}$$

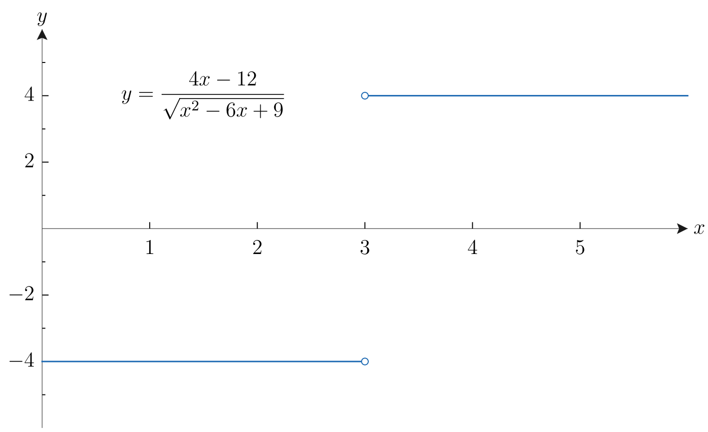
$$\lim_{\theta\to 3^+} \sqrt{\theta - 3} =$$

2. $\lim_{x\to 3} \frac{4x-12}{\sqrt{x^2-6x+9}} = \lim_{x\to 3} \frac{4(x-3)}{\sqrt{(x-3)^2}} = \lim_{x\to 3} \frac{4(x-3)}{|x-3|}$

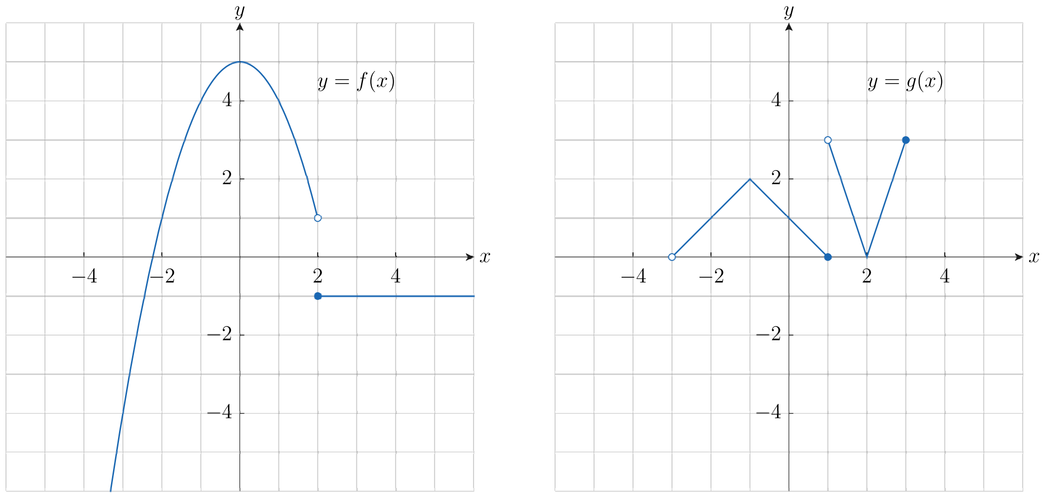
$$\lim_{x\to 3^+} \frac{4(x-3)}{|x-3|} = \lim_{x\to 3^+} \frac{4(x-3)}{x-3} = \lim_{x\to 3^+} 4 = 4$$

$$\lim_{x\to 3^-} \frac{4(x-3)}{|x-3|} = \lim_{x\to 3^-} \frac{4(x-3)}{-(x-3)} = \lim_{x\to 3^-} -4 = -4$$

$$\lim_{x\to 3} \frac{4x-12}{\sqrt{x^2-6x+9}} \text{ DNE}$$



3.



$$\lim_{x \to 2^-} g(f(x)) =$$

$$\lim_{x \to 2^+} g(f(x)) =$$

4. Consider the function

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 1 \\ x^2 & \text{if } 1 \le x < 2 \\ \sqrt{8x} & \text{if } 2 < x \le 8 \\ 8.0001 & \text{if } x > 8 \end{cases}$$

Determine the values of x for which the function f is continuous.

f is discontinuous at $x = 0$.

$x = 1$:

$f(1) = 1^2 = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x} = 1 \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1^1 = 1$

$\lim_{x\to 1} f(x) = 1 = f(1)$

$x = 2$:

$f(2)$ is not defined. Therefore f is not continuous at $x = 2$.

$x = 8$:

$f(8) = \sqrt{8 \cdot 8} = \sqrt{64} = 8$

$\lim_{x\to 8^-} f(x) = \lim_{x\to 8^-} \sqrt{8x} = \sqrt{64} = 8 \qquad \lim_{x\to 8^+} f(x) = \lim_{x\to 8^+} 8.0001 = 8.0001$

$\lim_{x\to 8} f(x)$ DNE; f is discontinuous at $x = 8$.

f is continuous everywhere but at $x = 0,\ 2,\ 8$

5. Let the functions u and r be defined by:

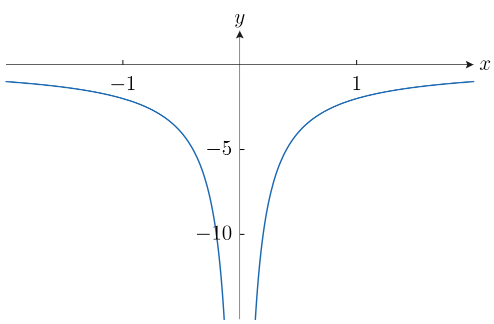
$$u(x) = \begin{cases} 2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -2 & \text{if } x > 0 \end{cases} \qquad r(x) = \begin{cases} 2 & \text{if } x = 0 \\ x & \text{if } x \neq 0 \end{cases}$$

Find $\displaystyle\lim_{x \to 0} \frac{u(x)}{r(x)}$

Solution

$$\lim_{x \to 0^-} \frac{u(x)}{r(x)} =$$

$$\lim_{x \to 0^+} \frac{u(x)}{r(x)} =$$



6. $\lim_{x \to 0} \dfrac{\sin x}{8x}$

Solution

$$\lim_{x \to 0} \frac{\sin x}{8x} = \frac{1}{8} \lim_{x \to 0} \frac{\sin x}{x}$$

Let $f(x) = \sin x$. Find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
&= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}
\end{aligned}$$

$\lim_{h \to 0} \sin x = \sin x$ and $\lim_{h \to 0} \cos x = \cos x$

$$\lim_{h \to 0} \frac{\sin h}{h}$$

Graphical, numerical evidence?

Geometric argument: $\cos h < \dfrac{\sin h}{h} < 1$

7. Let f be defined as

$$f(x) = \begin{cases} \sqrt{x-3} & \text{if } x > 3 \\ 6-2x & \text{if } x \leq 3 \end{cases}$$

Which of the following is true?

I. $\lim\limits_{x \to 3} \sqrt{x-3} = 0$

II. $\lim\limits_{x \to 3} (6-2x) = 0$

III. $\lim\limits_{x \to 3} f(x) = 0$