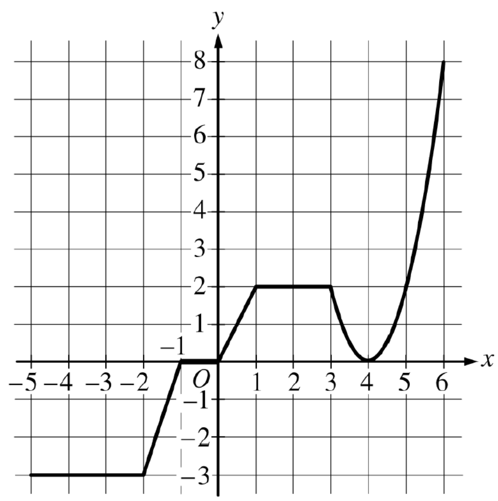
Monday Night Calculus, January 10, 2022

1. What happens if f' flatlines at 0 for a while? (Jen Spoerke)

2018 AB3



Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_1^6 g(x)\, dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Solution

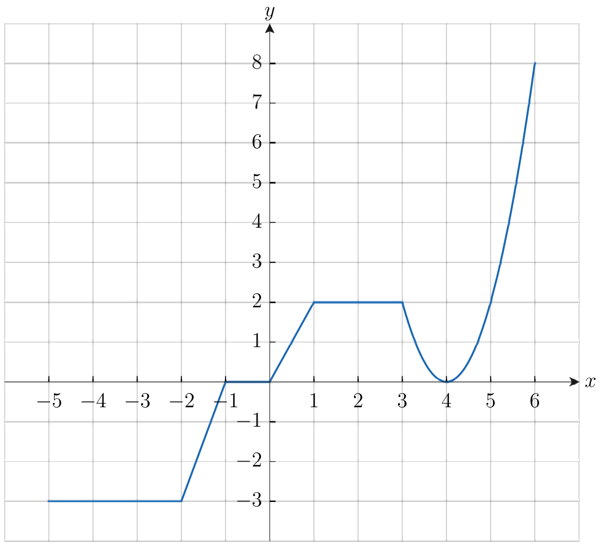
(a) $\displaystyle\int_{-5}^{1} g(x)\,dx = f(1) - f(-5)$

$$f(-5) = f(1) - \int_{-5}^{1} g(x)\,dx$$

$$= 3 - \left(-3\cdot 3 - \frac{1}{2}\cdot 1\cdot 3 + \frac{1}{2}\cdot 1\cdot 2\right)$$

$$= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$$

(b)



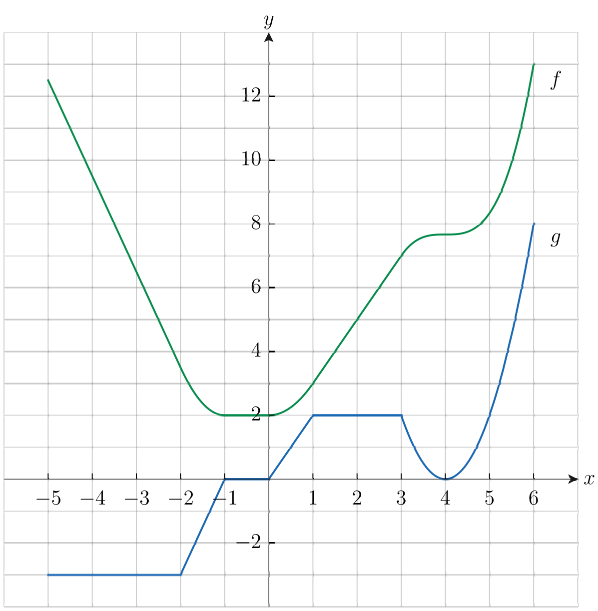
$$\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 2(x-4)^2\,dx$$

$$= 2 \cdot 2 + \left[\frac{2}{3}(x-4)^3\right]_3^6$$

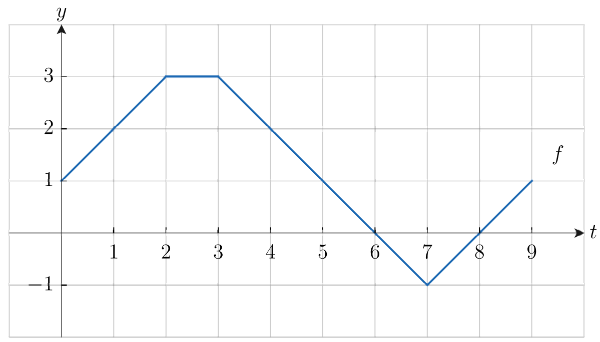
$$= 4 + \frac{2}{3}(8 - (-1)) = 4 + \frac{2}{3} \cdot 9 = 10$$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.

(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.



2. Let $g(x) = \int_a^x f(t)\,dt$ with the graph of f shown in the figure and a is a constant. Find the x-values of g regarding each of the following conditions. (Brendan Hughes)



(a) Relative minimum(s)

$g'(x) = f(x)$

g has a relative minimum at $x = 8$ because $g'(x) = f(x)$ changes from negative to positive there.

(b) Relative maximum(s)

g has a relative maximum at $x = 6$ because $g'(x) = f(x)$ changes from positive to negative there.

(c) Concave up

g is concave up on the intervals $(0, 2)$ and $(7, 9)$ because $g'(x) = f(x)$ is increasing on those intervals.

(d) Concave down

g is concave down on the interval $(3, 7)$ because $g'(x) = f(x)$ is decreasing on that interval.

(e) Increasing: $[1, 6]$, $[8, 9]$

Decreasing: $[6, 8]$

Points of inflection: $x = 7$

(f) If $g(2) = 1$, what is the maximum value of g on the interval $[2, 9]$?

$$g(6) = \int_a^6 f(t)\,dt = \int_a^2 f(t)\,dt + \int_2^6 f(t)\,dt$$

$$= 1 + 3 + \frac{1}{2} \cdot 3 \cdot 3 = \frac{17}{2}$$

(g) Suppose $h(x) = \int_0^{\frac{x}{2}+5} f(t)\,dt$. Find the x-value where h has a relative minimum.

$$h'(x) = f\left(\frac{x}{2}+5\right)\cdot\frac{1}{2} \qquad \frac{x}{2}+5=8 \;\Rightarrow\; x=6$$

3. $\displaystyle\int \frac{x\arctan x}{(1+x^2)^2}\,dx = I$ (Karen Martin Swift)

$$u = \arctan x \qquad dv = \frac{x}{(1+x^2)^2}\,dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = \int \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2(1+x^2)}$$

$$I = \arctan x \cdot \frac{-1}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{(1+x^2)^2}\,dx$$

$$x = \tan\theta \quad\Rightarrow\quad 1+x^2 = 1+\tan^2\theta = \sec^2\theta$$

$$dx = \sec^2\theta\,d\theta$$

$$\theta = \arctan x$$

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta = \int \cos^2\theta\,d\theta$$

$$= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

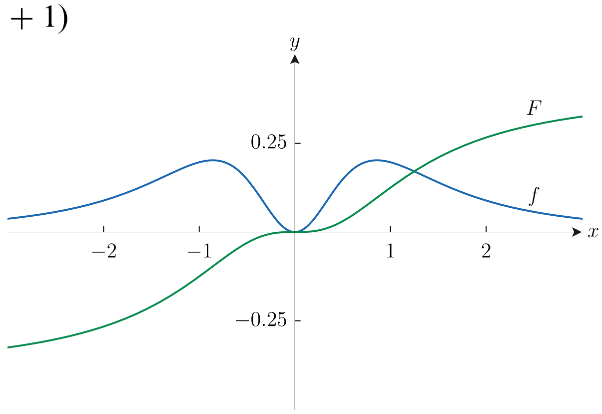
$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right)$$

$$I = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{2}\cdot\frac{1}{2}\left(\arctan x + \frac{x}{(1+x^2)}\right)$$

$$= -\frac{\arctan x}{2(1+x^2)} + \frac{\arctan x}{4} + \frac{x}{4(1+x^2)}$$

$$= \cdots = \frac{\left(x^2-1\right)\arctan(x) + x}{4\,(x^2+1)}$$



4. $\lim\limits_{x\to 1}\left[\dfrac{\ln x}{x^4-1}\right]^{1/2}$ (Carri Williams)

$$\lim_{x\to 1}\left[\frac{\ln x}{x^4-1}\right]^{1/2} \stackrel{?}{=} \left[\lim_{x\to 1}\frac{\ln x}{x^4-1}\right]^{1/2}$$

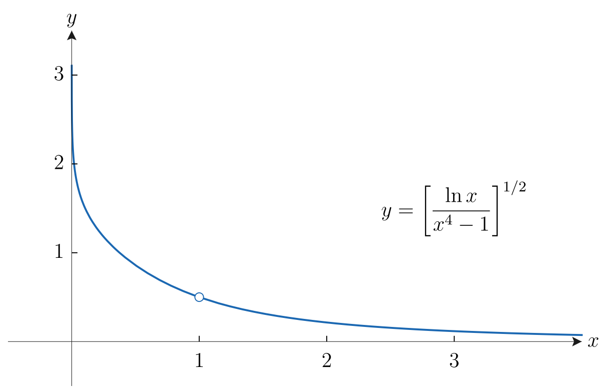
$\lim\limits_{x\to 1}\ln x = 0$ and $\lim\limits_{x\to 1}(x^4-1) = 0$

$$\lim_{x\to 1}\frac{\ln x}{x^4-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{4x^3}$$

$$= \lim_{x\to 1}\frac{1}{4x^4} = \frac{1}{4}$$

$$\lim_{x\to 1}\left[\frac{\ln x}{x^4-1}\right]^{1/2} = \left[\frac{1}{4}\right]^{1/2} = \frac{1}{2}$$

$f(x) = \left[\dfrac{\ln x}{x^4-1}\right]^{1/2}$ Domain:



5. Find the solution of the differential equation that satisfies the initial condition.

$$\frac{dy}{dx} = \frac{x \sin x}{y}, \quad y(0) = -1$$

$$y\,dy = x \sin x\,dx$$

$$\frac{y^2}{2} = -x \cos x + \sin x + C$$ Integration by parts

$$\frac{1}{2} = 0 + 0 + C \quad \Rightarrow \quad C = \frac{1}{2}$$ Use initial condition

$$y^2 = -2x \cos x + 2 \sin x + 1$$

$y = -\sqrt{1 - 2x \cos x + 2 \sin x}$ because $y(0) = -1 < 0$